

PICTURE OF THE MONTH

Early Summer Tornado Situation

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The pictures comprising the satellite mosaic (fig. 1) were taken on June 26, 1969. The mosaic clearly depicts three distinct synoptic regimes. A large high-pressure area dominates the weather along the East Coast, while a Low and an accompanying frontal system are responsible for

the severe weather in the Midwest (fig. 2). These systems and their areas of influence are clearly marked by the various cloud patterns.

The typical multilayered cloudiness associated with a low-pressure area can be seen at (A) in figure 1. The small

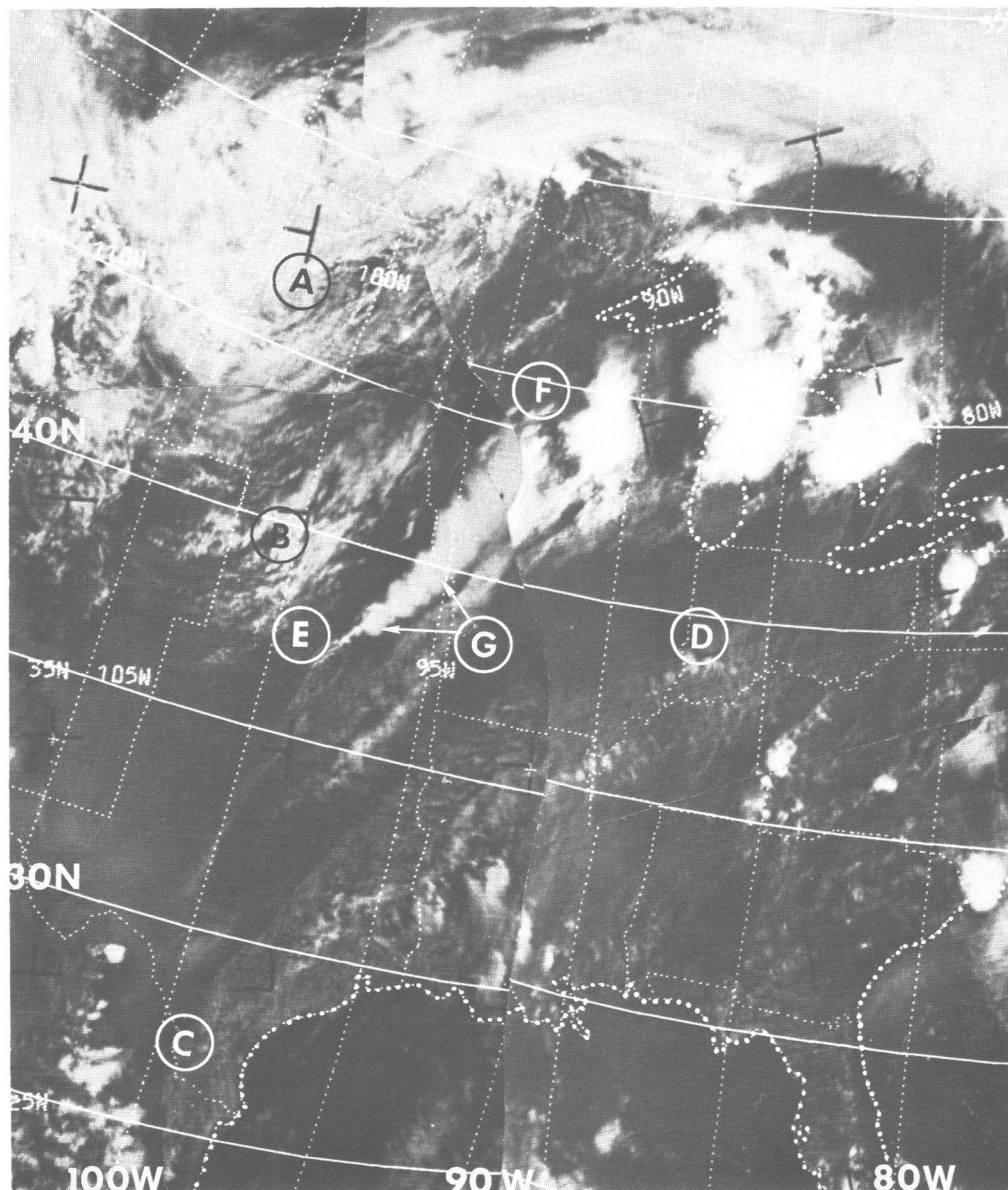


FIGURE 1.—Mosaic of ESSA 9 pictures, Passes 1504–1505, 1901–2057 GMT on June 26, 1969.

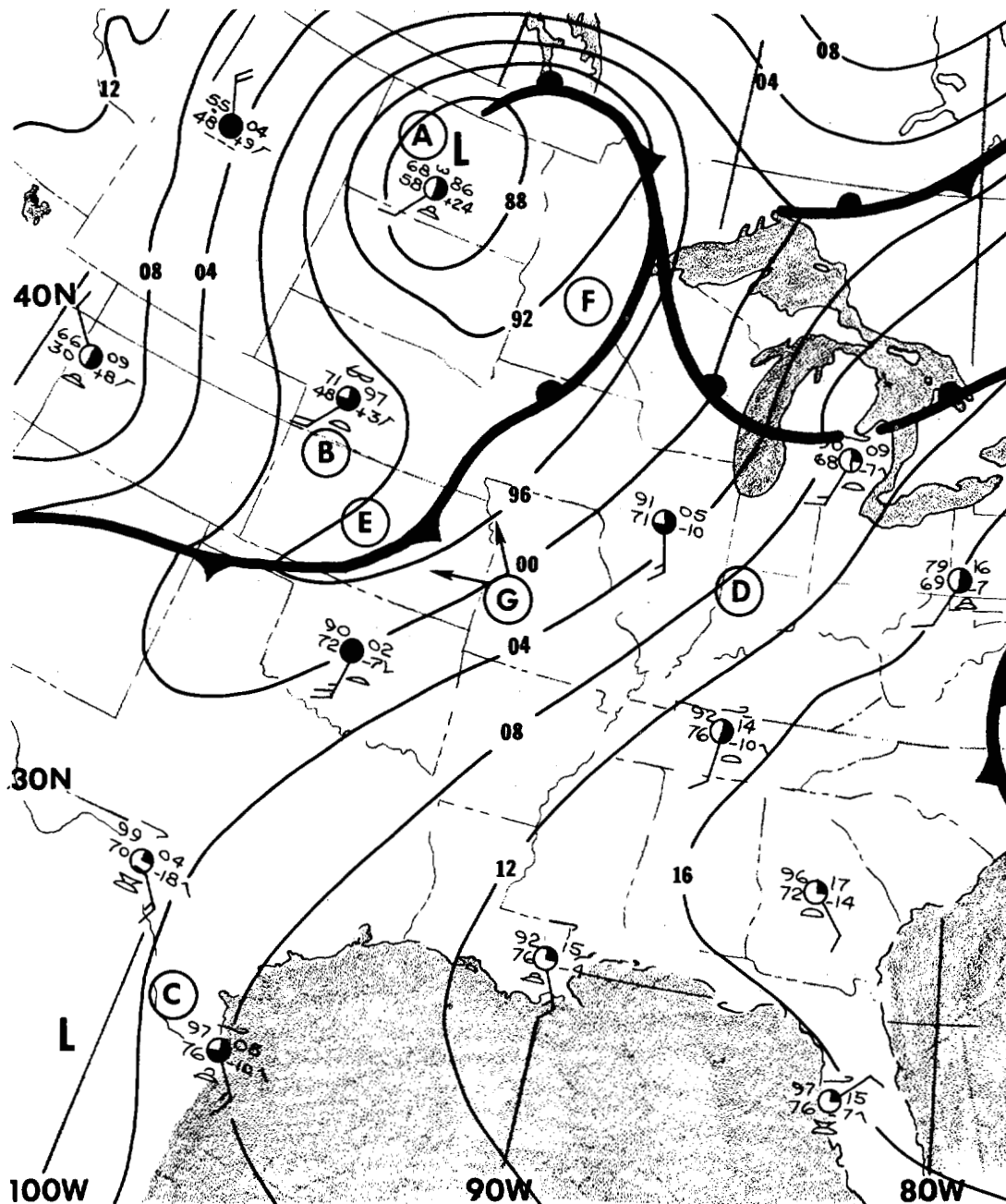


FIGURE 2.—Surface analysis, 2100 GMT on June 26, 1969.

area of cloudiness (B), south of the center, reflects the presence of a secondary vorticity center associated with this system.

An early morning APT (automatic picture transmission) view of this area showed low fog and stratus throughout the Gulf States. By the time of these ESSA 9 satellite pictures, heating had cleared the stratus; and a large area of fair-weather cumulus had formed. The northern edge of this cumulus field (C-D) lies along the edge of the lowest elevations of the Coastal Plain. Immediately west and parallel to this cumulus field is a long dark area of relatively clear skies ahead of the advancing frontal system.

The bright convective band (E-F) lies along the cold front and marks the interface of warm moist air from the south and the cold air associated with the advancing low-pressure area. Radar reports at 1945 GMT and 2045 GMT indicate that the tops of these convective cells extend from 45,000 to 63,000 ft. Most of the severe weather was confined to the southern portion of this cloud band (G). This convective cloud line has a shape that is often observed with severe weather outbreaks. The line is very narrow on its southwest or upwind end and broadens markedly to the northeast. Thirteen tornadoes and considerable wind and hail damage occurred in this area between 1900 and 2300 GMT.

CORRESPONDENCE

A Reliable Method for the Numerical Integration of a Large Class of Ordinary and Partial Differential Equations

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The purpose of this note is to describe the simple extension of a popular method of solving second-order ordinary differential equations with two end-point boundary conditions to n th order ordinary differential equations and to partial differential equations that are second order in one direction. The original method is simply a version of Gaussian elimination; and the extension (to be described) has, we have discovered, been published, slightly differently, before. We feel, however, that the present note will be of value, since the extension has proven very useful to both of us, and is seldom used among numerical analysts and meteorologists.

We begin by reviewing second-order ordinary differential equations. Consider

$$\frac{d^2f}{dx^2} + g(x) \frac{df}{dx} + h(x)f = r(x) \quad (1)$$

where

$$\frac{df}{dx} + a_1f = b_1 \text{ at } x=0$$

and

$$\frac{df}{dx} + a_2f = b_2 \text{ at } x=1.$$

In finite differences this becomes

$$A_nf_{n-1} + B_nf_n + C_nf_{n+1} = D_n, \quad (2)$$

$$n=1, 2, 3, \dots, N-1,$$

where

$$A_n = \frac{1}{(\delta x)^2} - \frac{g(x_n)}{2\delta x},$$

$$B_n = -\frac{2}{(\delta x)^2} + h(x_n),$$

$$C_n = \frac{1}{(\delta x)^2} + \frac{g(x_n)}{2\delta x},$$

and

$$D_n = r(x_n).$$

δx is the grid interval used in finite-difference approximation to equation (1),

$$A_b f_0 + B_b f_1 = D_b,$$

and

$$A_t f_{N-1} + B_t f_N = D_t$$

where N is the level number corresponding to $x=1$. The solution of equation (2) (following Richtmyer, 1957) goes as follows:

$$f_n = \alpha_n f_{n+1} + \beta_n \quad (3)$$

where α_n and β_n are newly introduced variables. Then

$$f_{n-1} = \alpha_{n-1} f_n + \beta_{n-1}. \quad (4)$$

Substituting equation (4) into (2) we obtain

$$\alpha_n = \frac{-C_n}{(A_n \alpha_{n-1} + B_n)} \quad (5)$$

and

$$\beta_n = \frac{(D_n - A_n \beta_{n-1})}{(A_n \alpha_{n-1} + B_n)}. \quad (6)$$

Thus, knowing α_0, β_0 we may readily obtain all α_n 's and β_n 's. From the lower boundary condition

$$\alpha_0 = -\frac{B_0}{A_0} \quad (7)$$

and

$$\beta_0 = \frac{D_0}{A_0}. \quad (8)$$

Equation (3) may be used to obtain f at all n 's, provided we know f_N . With the upper boundary condition we have

$$A_t f_{N-1} + B_t f_N = D_t. \quad (9)$$

We also have

$$f_{N-1} = \alpha_{N-1} f_N + \beta_{N-1}. \quad (10)$$

These may be solved to obtain

$$f_N = \frac{D_t - A_t \beta_{N-1}}{(B_t + \alpha_{N-1} A_t)}. \quad (11)$$

Thus our solution is formally complete. The procedure is valid provided that

$$A_n \alpha_{n-1} + B_n \neq 0$$

for all n . A sufficient condition for this to be so is that

$$0 < H_* \leq -h \leq H^*$$

and

$$\delta x \leq \frac{2}{G}$$

where $G = \max |g(x)|$; H_* and H^* are positive constants.

These are, however, by no means necessary conditions. The authors have yet to find an inhomogeneous, well-posed problem for which the method fails. In particular, many wave-type problems where $h > 0$ have been solved. It should be added that when $h = \text{constant}$ and $g = 0$, beyond a certain point in the domain, the requirement of two end points is readily extended to include a radiation condition. Let $h = \lambda^2$ for $x > x_1$. If we wish our solution to behave as $e^{i\lambda x}$ beyond x_1 , then we simply impose

$$df/dx = i\lambda f$$

at some $x > x_1$ as a boundary condition. Such an application may be found in Lindzen (1968). Also, when $h = -\lambda^2$ the method has no difficulty in separating growing from decaying solutions (Carrier and Pearson, 1968).

The extension of the above method to n th order ordinary differential equations is straightforward. Consider

$$\frac{d^n f}{dx^n} + g_1(x) \frac{d^{n-1} f}{dx^{n-1}} + g_2(x) \frac{d^{n-2} f}{dx^{n-2}} + \dots = r(x). \quad (12)$$

For simplicity let n be even. Also, let there be appropriate boundary conditions at $x=0$ and 1. What is meant by appropriate will become evident. Let

$$\begin{aligned} f_1 &= \frac{d^{n-2} f}{dx^{n-2}} \\ &\vdots \\ f_{\frac{n}{2}-1} &= \frac{d^2 f}{dx^2} \end{aligned} \quad (13a)$$

Equation (13a) may be rewritten

$$\frac{d^2 f_K}{dx^2} = f_{K-1}; \quad K=1, 2, \dots, m-1 \quad (13b)$$

where $m = n/2$. Equation (12) becomes

$$\frac{d^2 f_1}{dx^2} + g_1 \frac{df_1}{dx} + g_2 f_1 + \sum_{i=2}^m \left(g_{2i-1} \frac{df_i}{dx} + g_{2i} f_i \right) = r(x) \quad (14)$$

where $f_m = f$. Equations (13b) and (14) may be rewritten

$$\frac{d^2}{dx^2} \mathbf{f} + \mathcal{A}(x) \frac{d}{dx} \mathbf{f} + \mathcal{B}(x) \mathbf{f} = \mathbf{r}(x) \quad (15)$$

where

$$\mathbf{f} = (f_1, f_2, \dots, f_{\frac{n}{2}-1}, f)$$

and

$$\mathcal{A} = \begin{bmatrix} g_1 & g_3 & g_5 & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

$$\mathcal{B} = \begin{bmatrix} g_2 & g_4 & g_6 & \cdot & \cdot & \cdot & \cdot \\ -1 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & -1 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & -1 & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

$$\mathbf{r} = \begin{bmatrix} r \\ 0 \\ 0 \\ \cdot \\ \cdot \end{bmatrix}$$

Instead of equation (2), we now write

$$\mathbf{A}_n \mathbf{f}_{n-1} + \mathbf{B}_n \mathbf{f}_n + \mathbf{C}_n \mathbf{f}_{n+1} = \mathbf{D}_n \quad (16)$$

where equation (16) is the finite-difference approximation to (15);

$$\mathbf{A}_n = \frac{1}{(\delta x)^2} \mathbf{I} - \frac{1}{2\delta x} \mathcal{A}(x),$$

$$\mathbf{B}_n = -\frac{2}{(\delta x)^2} \mathbf{I} + \mathcal{B}(x),$$

$$\mathbf{C}_n = \frac{1}{(\delta x)^2} \mathbf{I} + \frac{1}{2\delta x} \mathcal{A}(x);$$

$$\mathbf{D}_n = \mathbf{r}(x_n)$$

and instead of equation (3), we write

$$\mathbf{f}_n = \alpha_n \mathbf{f}_{n+1} + \beta_n \quad (17)$$

where α_n is now an $(n/2) \times (n/2)$ matrix and β_n is as an n -dimensional vector. It is easily shown that

$$\alpha_n = -(\mathbf{A}_n \alpha_{n-1} + \mathbf{B}_n)^{-1} \mathbf{C}_n \quad (18)$$

and

$$\beta_n = (\mathbf{A}_n \alpha_{n-1} + \mathbf{B}_n)^{-1} (\mathbf{D}_n - \mathbf{A}_n \beta_{n-1}). \quad (19)$$

Thus, if we obtain α_0 and β_0 from our boundary condition at $x=0$, we may readily obtain all the other α_n 's and β_n 's. At each step, however, we must invert an $(n/2) \times (n/2)$ matrix. For $n \leq 8$, this is a trivial matter. Even for $n \gtrsim 180$, sparse routines (involving Gaussian elimination) are remarkably effective. As before, the value of \mathbf{f}_N is obtained from the upper boundary condition together with the equation

$$\mathbf{f}_{N-1} = \alpha_{N-1} \mathbf{f}_N + \beta_{N-1}.$$

It should be added that many high-order differential equations result from combining several lower order differentials. Thus, a set of second-order equations may present themselves in the course of analysis—prior to the derivation of the single n th order equation.

Although we are not normally interested in 180th order ordinary differential equations, the limit becomes quite meaningful when we come to partial differential equations.

Consider an equation of the form

$$\frac{\partial^2 f}{\partial x^2} + \mathcal{L}_{y,x}[f] = r(y, x) \quad (20)$$

where $\mathcal{L}_{y,x}$ is a differential operator of arbitrary order in y —but of no greater than first order in x . The finite-difference form of equation (20) is also given by (16) where, however, f_n is now the set of the values of f at the n th level in x at all the grid points in y . The boundary conditions at $x=0, 1$ (or any other two points) are introduced as before; the boundary conditions at $y=0, 1$ are included in A_n, B_n, C_n, D_n . The present method appears to be genuinely insensitive to equation (20)'s type. Several problems with mixed hyperbolic-elliptic equations have been solved with no difficulty. It is our impression that whenever equation (20) together with its boundary conditions has a continuous solution the present method will determine it. In this respect, our method appears superior to iterative procedures which usually fail for operators that are not purely elliptic. Our method is similar to those described by Cornock (1954), Karlqvist (1952), and Schechter (1960) in connection with the solution of particular partial differential equations. While the application of the method to high-order ordinary differential equations is obvious, we are not familiar with earlier references in this connection. The disadvantage of our method (minor for our purposes) is that it requires the inversion of N (where N =the number of levels) $J \times J$ matrices (where J is either the number of grid points in the y -direction, or—in the case of ordinary differential equations—one-half the order of the differential equation), and the storage of $N J \times J$ matrices and $N J$ -dimensional vectors for use in the backward sweep. In an elegant extension of the method described here, Schechter (1960) reduced the solution of the system of equations (16) to the inversion of a single $J \times J$ matrix. Schechter's method has, however, a serious disadvantage. As the number of levels increases, the condition number of the matrix to be inverted increases. If the equation to be inverted is hyperbolic over a significant part of its domain, the rise in condition number can be astronomical—the matrix becoming uninvertible for practical purposes. Thus, Schechter's method is type-sensitive.

The method described in this paper has been successfully used by the authors to investigate the propagation of planetary scale equatorial waves through shear zones

with and without critical levels, the propagation of internal gravity waves with arbitrary distributions of temperature, viscosity, conductivity, anisotropic ion drag, Newtonian cooling and thermal excitation, and the non-linear flows in the boundary layer of a vortex. The method has also been used by Matsuno (personal communication) to study the propagation of internal Rossby waves in an atmosphere with an arbitrary distribution of zonal wind with latitude and altitude. The results of all these calculations will be published separately. In each case, however, all the matrices to be inverted were of low condition number, and accurately and easily inverted using standard "share" routines.

As a final comment, we should state that many equations of the form of equation (20) are more efficiently solved by relaxation methods. Moreover, when equation (20)'s x and y dependence is separable, a common method of solution is to Fourier transform out one of the dependencies and use the present method for solving the resulting second-order ordinary differential equations. The virtue of the present method is not that it is the most efficient method, but that it appears to be generally reliable.

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